

# Generalised matrix multivariate $T$ -distribution

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## Abstract

Supposing Kotz-Riesz type I and II distributions and their corresponding independent univariate Riesz distributions the associated generalised matrix multivariate  $T$  distributions, termed matrix multivariate  $T$ -Riesz distributions are obtained. In addition, its various properties are studied. All these results are obtained for real normed division algebras.

## 1 Introduction

In many statistical models, as an alternative to the use of matrix multivariate normal distribution from the 80's it has been assumed a matrix multivariate elliptical distribution. Actually, the matrix multivariate elliptical distribution is a family of distributions that includes the matrix multivariate normal, contaminated normal, Pearson type II and VII, Kotz, Jensen-Logistic, power exponential and Bessel distributions, among others. These distributions have tails that are more or less weighted, and/or display a greater or smaller degree of kurtosis than the normal distribution, refer to Fang and Zhang (1990) and Gupta and Varga (1993).

In addition, matrix multivariate elliptical distributions are of great interest due to the next invariance property: Assume that  $\mathbf{X}$  is distributed according to a matrix multivariate distribution, then the distributions of certain type of matrix transformations of the random matrix, say  $\mathbf{Y} = f(\mathbf{X})$ , are invariant under all class of matrix multivariate elliptical distribution, furthermore, such distributions coincide when  $\mathbf{X}$  is normally assumed, see Fang and Zhang (1990) and Gupta and Varga (1993).

However, this invariance property is present when certain statistical (probabilistic) dependence is assumed. For example, if  $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$  has a matrix multivariate elliptical distribution, then  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are statistically dependent, observing that  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are proba-

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bilistically independent if  $\mathbf{X}$  has a matrix multivariate normal distribution, Gupta and Varga (1993). Then, if is defined  $\mathbf{T} = \mathbf{X}_1(\mathbf{X}_2'\mathbf{X}_2)^{-1/2}$ , it is said that  $\mathbf{T}$  has a matrix multivariate  $T$ -distribution, and its distribution is the same under all matrix multivariate elliptical distribution and this coincides with the distribution obtained when  $\mathbf{X}$  follow a matrix multivariate normal distribution.

The independent case cited above can be found in the Bayesian inference, see Press (1982). In particular, assume that certain distribution is function of two matrix parameters, say  $\delta_1, \delta_2$  for which, it is suppose that their prior distributions belong to the class of matrix variate elliptical distribution and are independent. Then, is of interest find the prior distribution of a parameter type  $\mathbf{T}$  defined as  $\delta_1(\delta_2\delta_2)^{-1/2}$ . In this case the distribution of  $\mathbf{T}$  is different for each particular elliptical distribution.

A distribution of particular interest is the matrix multivariate elliptical distribution termed Kotz-Riesz distribution. This interest is based in the relation with the Riesz distribution, Díaz-García (2013c). If  $\mathbf{X}$  is distributed according a matrix multivariate Kotz-Riesz, then the matrix  $\mathbf{V} = \mathbf{X}'\mathbf{X}$  has a Riesz distribution. The Riesz distributions, was first introduced by Hassairi and Lajmi (2001) under the name of Riesz natural exponential family (Riesz NEF); it was based on a special case of the so-called Riesz measure from Faraut and Korányi (1994, p.137). This Riesz distribution generalises the matrix multivariate gamma and Wishart distributions, containing them as particular cases.

In analogy with the case of  $T$ -distribution under normality, exist two possible generalisations of it when a Kotz-Riesz distribution is assumed, see Díaz-García and Gutiérrez-Jáimez (2012). In this paper is addressed the case of the distribution termed matrix multivariate  $T$ -Riesz distribution.

This present article is organised as follow; some basic concepts and the notation of abstract algebra and Jacobians are summarised in Section 2. The nonsingular central matrix multivariate  $T$ -Riesz type I and II distributions and the corresponding generalised beta type II distributions are studied in Section 3. Finally, the joint densities of the singular values are derived in Section 4. All these results are derived for real normed division algebras.

## 2 Preliminary results

A detailed discussion of real normed division algebras can be found in Baez (2002) and Neukirch *et al.* (1990). For your convenience, we shall introduce some notation, although in general, we adhere to standard notation forms.

For our purposes: Let  $\mathbb{F}$  be a field. An *algebra*  $\mathfrak{A}$  over  $\mathbb{F}$  is a pair  $(\mathfrak{A}; m)$ , where  $\mathfrak{A}$  is a *finite-dimensional vector space* over  $\mathbb{F}$  and *multiplication*  $m : \mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$  is an  $\mathbb{F}$ -bilinear map; that is, for all  $\lambda \in \mathbb{F}$ ,  $x, y, z \in \mathfrak{A}$ ,

$$\begin{aligned} m(x, \lambda y + z) &= \lambda m(x; y) + m(x; z) \\ m(\lambda x + y; z) &= \lambda m(x; z) + m(y; z). \end{aligned}$$

Two algebras  $(\mathfrak{A}; m)$  and  $(\mathfrak{E}; n)$  over  $\mathbb{F}$  are said to be *isomorphic* if there is an invertible map  $\phi : \mathfrak{A} \rightarrow \mathfrak{E}$  such that for all  $x, y \in \mathfrak{A}$ ,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write  $m(x; y) = xy$  for all  $x, y \in \mathfrak{A}$ .

Let  $\mathfrak{A}$  be an algebra over  $\mathbb{F}$ . Then  $\mathfrak{A}$  is said to be

1. *alternative* if  $x(xy) = (xx)y$  and  $x(yy) = (xy)y$  for all  $x, y \in \mathfrak{A}$ ,
2. *associative* if  $x(yz) = (xy)z$  for all  $x, y, z \in \mathfrak{A}$ ,

3. *commutative* if  $xy = yx$  for all  $x, y \in \mathfrak{A}$ , and
4. *unital* if there is a  $1 \in \mathfrak{A}$  such that  $x1 = x = 1x$  for all  $x \in \mathfrak{A}$ .

If  $\mathfrak{A}$  is unital, then the identity 1 is uniquely determined.

An algebra  $\mathfrak{A}$  over  $\mathbb{F}$  is said to be a *division algebra* if  $\mathfrak{A}$  is nonzero and  $xy = 0_{\mathfrak{A}} \Rightarrow x = 0_{\mathfrak{A}}$  or  $y = 0_{\mathfrak{A}}$  for all  $x, y \in \mathfrak{A}$ .

The term “division algebra”, comes from the proposition where, in such an algebra, left and right division can be unambiguously performed.

Let  $\mathfrak{A}$  be an algebra over  $\mathbb{F}$ . Then  $\mathfrak{A}$  is a division algebra if, and only if,  $\mathfrak{A}$  is nonzero and for all  $a, b \in \mathfrak{A}$ , with  $b \neq 0_{\mathfrak{A}}$ , the equations  $bx = a$  and  $yb = a$  have unique solutions  $x, y \in \mathfrak{A}$ .

In the sequel we assume  $\mathbb{F} = \mathbb{R}$  and consider classes of division algebras over  $\mathbb{R}$  or “*real division algebras*” for short.

We introduce the algebras of *real numbers*  $\mathbb{R}$ , *complex numbers*  $\mathbb{C}$ , *quaternions*  $\mathbb{H}$  and *octonions*  $\mathbb{O}$ . Then, if  $\mathfrak{A}$  is an alternative real division algebra, then  $\mathfrak{A}$  is isomorphic to  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  or  $\mathbb{O}$ .

Let  $\mathfrak{A}$  be a real division algebra with identity 1. Then  $\mathfrak{A}$  is said to be *normed* if there is an inner product  $(\cdot, \cdot)$  on  $\mathfrak{A}$  such that

$$(xy, xy) = (x, x)(y, y) \quad \text{for all } x, y \in \mathfrak{A}.$$

If  $\mathfrak{A}$  is a *real normed division algebra*, then  $\mathfrak{A}$  is isomorphic to  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  or  $\mathbb{O}$ .

There are exactly four normed division algebras: real numbers ( $\mathbb{R}$ ), complex numbers ( $\mathbb{C}$ ), quaternions ( $\mathbb{H}$ ) and octonions ( $\mathbb{O}$ ), see Baez (2002). We take into account that,  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{O}$  are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let  $\mathfrak{A}$  be a division algebra over the real numbers. Then  $\mathfrak{A}$  has dimension either 1, 2, 4 or 8. Other branches of mathematics used the parameters  $\alpha = 2/\beta$  and  $t = \beta/4$ , see Edelman and Rao (2005) and Kabe (1984), respectively.

Finally, observe that

- $\mathbb{R}$  is a real commutative associative normed division algebras,
- $\mathbb{C}$  is a commutative associative normed division algebras,
- $\mathbb{H}$  is an associative normed division algebras,
- $\mathbb{O}$  is an alternative normed division algebras.

Let  $\mathfrak{L}_{m,n}^{\beta}$  be the set of all  $m \times n$  matrices of rank  $m \leq n$  over  $\mathfrak{A}$  with  $m$  distinct positive singular values, where  $\mathfrak{A}$  denotes a *real finite-dimensional normed division algebra*. Let  $\mathfrak{A}^{m \times n}$  be the set of all  $m \times n$  matrices over  $\mathfrak{A}$ . The dimension of  $\mathfrak{A}^{m \times n}$  over  $\mathbb{R}$  is  $\beta mn$ . Let  $\mathbf{A} \in \mathfrak{A}^{m \times n}$ , then  $\mathbf{A}^* = \mathbf{A}^T$  denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

Table 1: Notation				
Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}_{m,n}^{\beta}$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^{\beta}(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	$\mathfrak{S}_m^{\beta}$

It is denoted by  $\mathfrak{S}_m^{\beta}$  the real vector space of all  $\mathbf{S} \in \mathfrak{A}^{m \times m}$  such that  $\mathbf{S} = \mathbf{S}^*$ . In addition, let  $\mathfrak{P}_m^{\beta}$  be the *cone of positive definite matrices*  $\mathbf{S} \in \mathfrak{A}^{m \times m}$ . Thus,  $\mathfrak{P}_m^{\beta}$  consist of all matrices  $\mathbf{S} = \mathbf{X}\mathbf{X}^*$ , with  $\mathbf{X} \in \mathfrak{L}_{m,n}^{\beta}$ ; then  $\mathfrak{P}_m^{\beta}$  is an open subset of  $\mathfrak{S}_m^{\beta}$ .

Let  $\mathfrak{D}_m^\beta$  be the *diagonal subgroup* of  $\mathcal{L}_{m,m}^\beta$  consisting of all  $\mathbf{D} \in \mathfrak{A}^{m \times m}$ ,  $\mathbf{D} = \text{diag}(d_1, \dots, d_m)$ . Let  $\mathfrak{T}_U^\beta(m)$  be the subgroup of all *upper triangular* matrices  $\mathbf{T} \in \mathfrak{A}^{m \times m}$  such that  $t_{ij} = 0$  for  $1 < i > j \leq m$ .

The set of matrices  $\mathbf{H}_1 \in \mathfrak{F}^{m \times n}$  such that  $\mathbf{H}_1 \mathbf{H}_1^* = \mathbf{I}_m$  is a manifold denoted  $\mathcal{V}_{m,n}^\beta$ , is termed the *Stiefel manifold* ( $\mathbf{H}_1$  is also known as *semi-orthogonal*).

For any matrix  $\mathbf{X} \in \mathfrak{A}^{m \times n}$ ,  $d\mathbf{X}$  denotes the *matrix of differentials*  $(dx_{ij})$ . Finally, we define the *measure* or volume element  $(d\mathbf{X})$  when  $\mathbf{X} \in \mathfrak{A}^{m \times n}, \mathfrak{S}_m^\beta, \mathfrak{D}_m^\beta$  or  $\mathcal{V}_{m,n}^\beta$ , see Díaz-García and Gutiérrez-Jáimez (2011).

If  $\mathbf{X} \in \mathfrak{A}^{m \times n}$  then  $(d\mathbf{X})$  (the Lebesgue measure in  $\mathfrak{A}^{m \times n}$ ) denotes the exterior product of the  $\beta mn$  functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^m \bigwedge_{j=1}^n dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^\beta dx_{ij}^{(k)}.$$

If  $\mathbf{S} \in \mathfrak{S}_m^\beta$  (or  $\mathbf{S} \in \mathfrak{T}_U^\beta(m)$  with  $t_{ii} > 0$ ,  $i = 1, \dots, m$ ) then  $(d\mathbf{S})$  (the Lebesgue measure in  $\mathfrak{S}_m^\beta$  or in  $\mathfrak{T}_U^\beta(m)$ ) denotes the exterior product of the  $m(m-1)\beta/2 + m$  functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^m ds_{ii} \bigwedge_{i < j}^m \bigwedge_{k=1}^\beta ds_{ij}^{(k)}.$$

Observe, that for the Lebesgue measure  $(d\mathbf{S})$ , it is required that  $\mathbf{S} \in \mathfrak{P}_m^\beta$ , that is,  $\mathbf{S}$  must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If  $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$  then  $(d\mathbf{\Lambda})$  (the Lebesgue measure in  $\mathfrak{D}_m^\beta$ ) denotes the exterior product of the  $\beta m$  functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^n \bigwedge_{k=1}^\beta d\lambda_i^{(k)}.$$

If  $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$  is such that  $\mathbf{H}_1 = (\mathbf{h}_1^*, \dots, \mathbf{h}_m^*)^*$ , where  $\mathbf{h}_i$ ,  $i = 1, \dots, m$  are their rows, then

$$(\mathbf{H}_1 d\mathbf{H}_1^*) = \bigwedge_{i=1}^m \bigwedge_{j=i+1}^n \mathbf{h}_j^* d\mathbf{h}_i, \quad (1)$$

where the partitioned matrix  $\mathbf{H} = (\mathbf{H}_1^* | \mathbf{H}_2^*)^* = (\mathbf{h}_1^*, \dots, \mathbf{h}_m^* | \mathbf{h}_{m+1}^*, \dots, \mathbf{h}_n^*)^* \in \mathfrak{U}^\beta(n)$ , with  $\mathbf{H}_2 = (\mathbf{h}_{m+1}, \dots, \mathbf{h}_n)$ . It can be proved that this differential form does not depend on the choice of the  $\mathbf{H}_2$  matrix. When  $n = 1$ ;  $\mathcal{V}_{m,1}^\beta$  defines the unit sphere in  $\mathfrak{A}^m$ . This is, of course, an  $(m-1)\beta$ - dimensional surface in  $\mathfrak{A}^m$ .

The surface area or volume of the Stiefel manifold  $\mathcal{V}_{m,n}^\beta$  is

$$\text{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathbf{H}_1 d\mathbf{H}_1^*) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \quad (2)$$

where  $\Gamma_m^\beta[a]$  denotes the multivariate *Gamma function* for the space  $\mathfrak{S}_m^\beta$ . This can be obtained as a particular case of the *generalised gamma function of weight  $\kappa$*  for the space  $\mathfrak{S}_m^\beta$  with  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ , taking  $\kappa = (0, 0, \dots, 0)$  and which

for  $\text{Re}(a) \geq (m-1)\beta/2 - k_m$  is defined by, see Gross and Richards (1987).

$$\Gamma_m^\beta[a, \kappa] = \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}) (d\mathbf{A}) \quad (3)$$

$$\begin{aligned} &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2] \\ &= [a]_\kappa^\beta \Gamma_m^\beta[a], \end{aligned} \quad (4)$$

where  $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$ ,  $|\cdot|$  denotes the determinant, and for  $\mathbf{A} \in \mathfrak{S}_m^\beta$

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}} \quad (5)$$

with  $\mathbf{A}_p = (a_{rs})$ ,  $r, s = 1, 2, \dots, p$ ,  $p = 1, 2, \dots, m$  is termed the *highest weight vector*, see Gross and Richards (1987). Also,

$$\begin{aligned} \Gamma_m^\beta[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - (i-1)\beta/2], \end{aligned}$$

and  $\text{Re}(a) > (m-1)\beta/2$ .

In other branches of mathematics the *highest weight vector*  $q_\kappa(\mathbf{A})$  is also termed the *generalised power* of  $\mathbf{A}$  and is denoted as  $\Delta_\kappa(\mathbf{A})$ , see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Additional properties of  $q_\kappa(\mathbf{A})$ , which are immediate consequences of the definition of  $q_\kappa(\mathbf{A})$  and the following property 1, are:

1. if  $\lambda_1, \dots, \lambda_m$ , are the eigenvalues of  $\mathbf{A}$ , then

$$q_\kappa(\mathbf{A}) = \prod_{i=1}^m \lambda_i^{k_i}. \quad (6)$$

- 2.

$$q_\kappa(\mathbf{A}^{-1}) = q_\kappa^{-1}(\mathbf{A}) = q_{-\kappa}(\mathbf{A}), \quad (7)$$

3. if  $\kappa = (p, \dots, p)$ , then

$$q_\kappa(\mathbf{A}) = |\mathbf{A}|^p, \quad (8)$$

in particular if  $p = 0$ , then  $q_\kappa(\mathbf{A}) = 1$ .

4. if  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ , then

$$q_{\kappa+\tau}(\mathbf{A}) = q_\kappa(\mathbf{A}) q_\tau(\mathbf{A}), \quad (9)$$

in particular if  $\tau = (p, p, \dots, p)$ , then

$$q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_\kappa(\mathbf{A}). \quad (10)$$

5. Finally, for  $\mathbf{B} \in \mathfrak{A}^{m \times m}$  in such a manner that  $\mathbf{C} = \mathbf{B}^* \mathbf{B} \in \mathfrak{S}_m^\beta$ ,

$$q_\kappa(\mathbf{B} \mathbf{A} \mathbf{B}^*) = q_\kappa(\mathbf{C}) q_\kappa(\mathbf{A}) \quad (11)$$

and

$$q_\kappa(\mathbf{B}^{-1} \mathbf{A} \mathbf{B}^{*-1}) = (q_\kappa(\mathbf{C}))^{-1} q_\kappa(\mathbf{A}). \quad (12)$$

**Remark 2.1.** Let  $\mathcal{P}(\mathfrak{S}_m^\beta)$  denote the algebra of all polynomial functions on  $\mathfrak{S}_m^\beta$ , and  $\mathcal{P}_k(\mathfrak{S}_m^\beta)$  the subspace of homogeneous polynomials of degree  $k$  and let  $\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$  be an irreducible subspace of  $\mathcal{P}(\mathfrak{S}_m^\beta)$  such that

$$\mathcal{P}_k(\mathfrak{S}_m^\beta) = \sum_{\kappa} \bigoplus \mathcal{P}^\kappa(\mathfrak{S}_m^\beta).$$

Note that  $q_\kappa$  is a homogeneous polynomial of degree  $k$ , moreover  $q_\kappa \in \mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$ , see Gross and Richards (1987).  $\square$

In (4),  $[a]_\kappa^\beta$  denotes the generalised Pochhammer symbol of weight  $\kappa$ , defined as

$$\begin{aligned} [a]_\kappa^\beta &= \prod_{i=1}^m (a - (i-1)\beta/2)_{k_i} \\ &= \frac{\pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2]}{\Gamma_m^\beta[a]} \\ &= \frac{\Gamma_m^\beta[a, \kappa]}{\Gamma_m^\beta[a]}, \end{aligned}$$

where  $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$  and

$$(a)_i = a(a+1) \cdots (a+i-1),$$

is the standard Pochhammer symbol.

An alternative definition of the generalised gamma function of weight  $\kappa$  is proposed by Khatri (1966), which is defined as

$$\Gamma_m^\beta[a, -\kappa] = \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}^{-1}) (d\mathbf{A}) \quad (13)$$

$$\begin{aligned} &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - k_i - (m-i)\beta/2] \\ &= \frac{(-1)^k \Gamma_m^\beta[a]}{[-a + (m-1)\beta/2 + 1]_\kappa^\beta}, \end{aligned} \quad (14)$$

where  $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$ .

In addition consider the following generalised beta functions, see Faraut and Korányi (1994, p. 130) and Díaz-García (2013b),

$$\begin{aligned} \mathcal{B}_m^\beta[a, \kappa; b, \tau] &= \int_{\mathbf{0} < \mathbf{S} < \mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S}) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{I}_m - \mathbf{S}) (d\mathbf{S}) \\ &= \int_{\mathbf{R} \in \mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{\kappa+\tau}^{-1}(\mathbf{I}_m + \mathbf{R}) (d\mathbf{R}) \\ &= \frac{\Gamma_m^\beta[a, \kappa] \Gamma_m^\beta[b, \tau]}{\Gamma_m^\beta[a+b, \kappa+\tau]}, \end{aligned} \quad (15)$$

where  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers,  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers,  $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$  and  $\operatorname{Re}(b) > (m-1)\beta/2 - t_m$ . Similarly,

$$\mathcal{B}_m^\beta[a, -\kappa; b, -\tau]$$

$$\begin{aligned}
&= \int_{\mathbf{0} < \mathbf{S} < \mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa^{-1}(\mathbf{S}) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau^{-1}(\mathbf{I}_m - \mathbf{S}) (d\mathbf{S}) \\
&= \int_{\mathbf{R} \in \mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa^{-1}(\mathbf{R}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{\kappa+\tau}(\mathbf{I}_m + \mathbf{R}) (d\mathbf{R}) \\
&= \frac{\Gamma_m^\beta[a, -\kappa] \Gamma_m^\beta[b, -\tau]}{\Gamma_m^\beta[a+b, -\kappa-\tau]},
\end{aligned}$$

where  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers,  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers,  $\text{Re}(a) > (m-1)\beta/2 + k_1$  and  $\text{Re}(b) > (m-1)\beta/2 + t_1$ .

Finally, four Jacobians in terms of the  $\beta$  parameter, which are proposed as extensions of real, complex or quaternion cases, see Díaz-García and Gutiérrez-Jáimez (2011).

**Proposition 2.1.** *Let  $\mathbf{X}$  and  $\mathbf{Y} \in \mathcal{L}_{m,n}^\beta$  be matrices of functionally independent variables, and let  $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{B} + \mathbf{C}$ , where  $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$ ,  $\mathbf{B} \in \mathcal{L}_{n,n}^\beta$  and  $\mathbf{C} \in \mathcal{L}_{m,n}^\beta$  are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^* \mathbf{A}|^{\beta n/2} |\mathbf{B}^* \mathbf{B}|^{\beta m/2} (d\mathbf{X}). \quad (16)$$

**Proposition 2.2.** *Let  $\mathbf{X}$  and  $\mathbf{Y} \in \mathfrak{S}_m^\beta$  be matrices of functionally independent variables, and let  $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{C}$ , where  $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$  and  $\mathbf{C} \in \mathfrak{S}_m^\beta$  are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^* \mathbf{A}|^{\beta(m-1)/2+1} (d\mathbf{X}). \quad (17)$$

**Proposition 2.3** (Singular Value Decomposition, *SVD*). *Let  $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$  be matrix of functionally independent variables, such that  $\mathbf{X} = \mathbf{W}^* \mathbf{D} \mathbf{V}_1$  with  $\mathbf{V}_1 \in \mathcal{V}_{m,n}^\beta$ ,  $\mathbf{W} \in \mathfrak{U}^\beta(m)$  and  $\mathbf{D} = \text{diag}(d_1, \dots, d_m) \in \mathfrak{D}_m^1$ ,  $d_1 > \dots > d_m > 0$ . Then*

$$(d\mathbf{X}) = 2^{-m} \pi^\tau \prod_{i=1}^m d_i^{\beta(n-m+1)-1} \prod_{i < j}^m (d_i^2 - d_j^2)^\beta (d\mathbf{D}) (\mathbf{V}_1 d\mathbf{V}_1^*) (\mathbf{W} d\mathbf{W}^*), \quad (18)$$

where

$$\tau = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

**Proposition 2.4.** *Let  $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$  be matrix of functionally independent variables, and  $\mathbf{S} = \mathbf{X}\mathbf{X}^* \in \mathfrak{P}_m^\beta$ . Then*

$$(d\mathbf{X}) = 2^{-m} |\mathbf{S}|^{\beta(n-m+1)/2-1} (d\mathbf{S}) (\mathbf{V}_1 d\mathbf{V}_1^*), \quad (19)$$

with  $\mathbf{V}_1 \in \mathcal{V}_{m,n}^\beta$ .

### 3 Matrix multivariate $T$ -Riesz distribution

A detailed discussion of Riesz distribution may be found in Hassairi and Lajmi (2001) and Díaz-García (2013a). In addition the Kotz-Riesz distribution is studied in detail in Díaz-García (2013c). For your convenience, we adhere to standard notation stated in Díaz-García (2013a,c).

This way, in this section, two versions of the matrix multivariate  $T$ -Riesz distribution and the corresponding generalised beta type II distributions are obtained.

**Theorem 3.1.** Let  $(S^{1/2})^2 = S \sim \mathcal{R}_1^{\beta,I}(\nu\beta/2, k, \rho)$ ,  $\rho > 0$ ,  $k$  is nonnegative integer and  $\text{Re}(\nu\beta/2) > -k$ ; independent of  $\mathbf{Y} \sim \mathcal{KR}_{m \times n}^{\beta,I}(\tau, \mathbf{0}, \mathbf{\Sigma}, \mathbf{\Theta})$ ,  $\mathbf{\Sigma} \in \mathfrak{P}_m^\beta$ ,  $\mathbf{\Theta} \in \mathfrak{P}_n^\beta$  and  $\text{Re}([n\beta/2] > (m-1)\beta/2 - t_m$ . In addition, define  $\mathbf{T}_1 = S^{-1/2}\mathbf{Y} + \boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$ . Then the density of  $\mathbf{T}_1$  is

$$\begin{aligned} & \propto [1 + \rho \text{tr } \mathbf{\Theta}^{-1}(\mathbf{T}_1 - \boldsymbol{\mu})^* \mathbf{\Sigma}^{-1}(\mathbf{T}_1 - \boldsymbol{\mu})]^{-(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i} \\ & \times q_\tau \left( \mathbf{\Theta}^{-1/2}(\mathbf{T}_1 - \boldsymbol{\mu})^* \mathbf{\Sigma}^{-1}(\mathbf{T}_1 - \boldsymbol{\mu}) \mathbf{\Theta}^{-1/2} \right) \end{aligned} \quad (20)$$

with constant of proportionality

$$\frac{\Gamma_m^\beta[n\beta/2] \Gamma_1^\beta[(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i] \rho^{\beta mn/2+\sum_{i=1}^m t_i}}{\pi^{\beta mn/2} \Gamma_m^\beta[n\beta/2, \tau] \Gamma_1^\beta[\nu\beta/2+k] |\mathbf{\Sigma}|^{\beta n/2} |\mathbf{\Theta}|^{\beta m/2}},$$

which is termed the matrix multivariate  $T$ -Riesz type I distribution and is denoted as  $\mathbf{T}_1 \sim \mathcal{MTR}_{m \times n}^{\beta,I}(\nu, k, \tau, \rho, \boldsymbol{\mu}, \mathbf{\Sigma}, \mathbf{\Theta})$ .

*Proof.* From Díaz-García (2013a,c), the joint density of  $S$  and  $\mathbf{Y}$  is

$$\propto s^{\beta\nu/2+k-1} \text{etr} \left\{ -\beta \left( s/\rho + \text{tr } \mathbf{\Theta}^{-1} \mathbf{Y}^* \mathbf{\Sigma}^{-1} \mathbf{Y} \right) \right\} q_\tau \left( \text{tr } \mathbf{\Theta}^{-1} \mathbf{Y}^* \mathbf{\Sigma}^{-1} \mathbf{Y} \right) (ds)(d\mathbf{Y})$$

where the constant of proportionality is

$$c = \frac{\beta^{\nu\beta/2+k}}{\Gamma_1^\beta[\nu\beta/2+k] \rho^{\nu\beta/2+k}} \cdot \frac{\beta^{mn\beta/2+\sum_{i=1}^m t_i} \Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2} \Gamma_m^\beta[n\beta/2, \tau] |\mathbf{\Sigma}|^{n\beta/2} |\mathbf{\Theta}|^{m\beta/2}}.$$

Taking into account that by (16)

$$(ds)(d\mathbf{Y}) = s^{\beta mn/2} (ds)(d\mathbf{T}_1),$$

the desired result is obtained integrating with respect to  $s$ .  $\square$   $\square$

Similarly is obtained:

**Theorem 3.2.** Let  $\mathbf{T}_1 = S^{-1/2}\mathbf{Y} + \boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$  where  $(S^{1/2})^2 = S \sim \mathcal{R}_1^{\beta,II}(\nu\beta/2, k, \rho)$ ,  $\rho > 0$ ,  $k$  is nonnegative integer and  $\text{Re}(\nu\beta/2) > k$ ; independent of  $\mathbf{Y} \sim \mathcal{KR}_{m \times n}^{\beta,II}(\tau, \mathbf{0}, \mathbf{\Sigma}, \mathbf{\Theta})$ ,  $\mathbf{\Sigma} \in \mathfrak{P}_m^\beta$ ,  $\mathbf{\Theta} \in \mathfrak{P}_n^\beta$  and  $\text{Re}([n\beta/2] > (m-1)\beta/2 - t_1$ . Then the density of  $\mathbf{T}_1$  is

$$\begin{aligned} & \propto [1 + \rho \text{tr } \mathbf{\Theta}^{-1}(\mathbf{T}_1 - \boldsymbol{\mu})^* \mathbf{\Sigma}^{-1}(\mathbf{T}_1 - \boldsymbol{\mu})]^{-(\nu+mn)\beta/2-k-\sum_{i=1}^m t_i} \\ & \times q_\tau^{-1} \left( \mathbf{\Theta}^{-1/2}(\mathbf{T}_1 - \boldsymbol{\mu})^* \mathbf{\Sigma}^{-1}(\mathbf{T}_1 - \boldsymbol{\mu}) \mathbf{\Theta}^{-1/2} \right) \end{aligned} \quad (21)$$

with constant of proportionality

$$\frac{\Gamma_m^\beta[n\beta/2] \Gamma_1^\beta[(\nu+mn)\beta/2-k-\sum_{i=1}^m t_i] \rho^{\beta mn/2-\sum_{i=1}^m t_i}}{\pi^{\beta mn/2} \Gamma_m^\beta[n\beta/2, -\tau] \Gamma_1^\beta[\nu\beta/2-k] |\mathbf{\Sigma}|^{\beta n/2} |\mathbf{\Theta}|^{\beta m/2}},$$

which is termed the matrix multivariate  $T$ -Riesz type II distribution and is denoted as  $\mathbf{T}_1 \sim \mathcal{MTR}_{m \times n}^{\beta,II}(\nu, k, \tau, \rho, \boldsymbol{\mu}, \mathbf{\Sigma}, \mathbf{\Theta})$ .

Next we study the corresponding matrix multivariate beta type II distributions.

**Theorem 3.3.** Define  $\mathbf{F}_1 = \mathbf{T}_1 \mathbf{T}_1^* \in \mathfrak{P}_m^\beta$ , with  $n \geq m$  and observe that

$$\mathbf{F}_1 = S^{-1} \mathbf{Y} \mathbf{Y}^* = S^{-1} \mathbf{W}.$$



1. If  $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^{\beta, I}(\nu, k, \tau, \rho, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ , then, under the conditions of Theorem 3.1 we have that,  $\mathbf{W} = \mathbf{Y}\mathbf{Y}^* \sim \mathcal{R}_m^{\beta, I}(n\beta/2, \tau, \mathbf{I}_m)$ , with  $\text{Re}(n\beta/2) > (m-1)\beta/2 - t_m$  and the density of  $\mathbf{F}_1$  is,

$$\propto |\mathbf{F}_1|^{(n-m+1)\beta/2-1} (1 + \rho \text{tr } \mathbf{F}_1)^{-[(mn+\nu)\beta/2+k+\sum_{i=1}^m t_i]} q_\tau(\mathbf{F}_1)(d\mathbf{F}_1), \quad (22)$$

with constant of proportionality

$$\frac{\Gamma_1^\beta[(\nu + mn)\beta/2 + k + \sum_{i=1}^m t_i] \rho^{\beta mn/2 + \sum_{i=1}^m t_i}}{\Gamma_m^\beta[n\beta/2, \tau] \Gamma_1^\beta[\nu\beta/2 + k]},$$

where  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 - k_m$  and  $\text{Re}(m\beta/2) > (m-1)\beta/2 - t_m$ .  $\mathbf{F}_1$  is said to have a matrix multivariate c-beta-Riesz type II distribution.

2. If  $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^{\beta, II}(\nu, k, \tau, \rho, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ , then, under the conditions of Theorem 3.2 we obtain that,  $\mathbf{W} = \mathbf{Y}\mathbf{Y}^* \sim \mathcal{R}_m^{\beta, II}(n\beta/2, \tau, \mathbf{I}_m)$ , with  $\text{Re}(n\beta/2) > (m-1)\beta/2 + t_1$  and the density of  $\mathbf{F}_1$  is,

$$\propto |\mathbf{F}_1|^{(n-m+1)\beta/2-1} (1 + \rho \text{tr } \mathbf{F}_1)^{-[(mn+\nu)\beta/2-k-\sum_{i=1}^m t_i]} q_\tau^{-1}(\mathbf{F}_1)(d\mathbf{F}_1), \quad (23)$$

with constant of proportionality

$$\frac{\Gamma_1^\beta[(\nu + mn)\beta/2 - k - \sum_{i=1}^m t_i] \rho^{\beta mn/2 - \sum_{i=1}^m t_i}}{\Gamma_m^\beta[n\beta/2, -\tau] \Gamma_1^\beta[\nu\beta/2 - k]},$$

where  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 + k_1$  and  $\text{Re}(m\beta/2) > (m-1)\beta/2 + t_1$ .  $\mathbf{F}_1$  is said to have a matrix multivariate k-beta-Riesz type II distribution.

*Proof.* The desired result follows from (20) and (21) respectively, by applying (19) and then (2).  $\square$   $\square$

Similarly, if  $n < m$  and  $\tilde{\mathbf{F}}_1 = \mathbf{T}_1^* \mathbf{T}_1 \in \mathfrak{P}_n^\beta$ .

**Theorem 3.4.** Observe that

$$\tilde{\mathbf{F}}_1 = S^{-1} \mathbf{Y}^* \mathbf{Y} = S^{-1} \mathbf{W}.$$

1. Then if  $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^{\beta, I}(\nu, k, \tau, \rho, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ , then, under the conditions of Theorem 3.1 we have that,  $\mathbf{W} = \mathbf{Y}^* \mathbf{Y} \sim \mathcal{R}_n^{\beta, I}(m\beta/2, \tau, \mathbf{I}_n)$ , with  $\text{Re}(m\beta/2) > (n-1)\beta/2 - t_n$  and the density of  $\tilde{\mathbf{F}}_1$  is,

$$\propto |\tilde{\mathbf{F}}_1|^{(m-n+1)\beta/2-1} (1 + \rho \text{tr } \tilde{\mathbf{F}}_1)^{-[(mn+\nu)\beta/2+k+\sum_{i=1}^n t_i]} q_\tau(\tilde{\mathbf{F}}_1)(d\tilde{\mathbf{F}}_1), \quad (24)$$

with constant of proportionality

$$\frac{\Gamma_1^\beta[(\nu + mn)\beta/2 + k + \sum_{i=1}^n t_i] \rho^{\beta mn/2 + \sum_{i=1}^n t_i}}{\Gamma_n^\beta[m\beta/2, \tau] \Gamma_1^\beta[\nu\beta/2 + k]},$$

where  $\text{Re}(\nu\beta/2) > (n-1)\beta/2 - k_n$  and  $\text{Re}(m\beta/2) > (n-1)\beta/2 - t_n$ .  $\tilde{\mathbf{F}}_1$  is said to have a matrix multivariate c-beta-Riesz type II distribution.

2. Then if  $\mathbf{T}_1 \sim \mathcal{MT}_{m \times n}^{\beta, II}(\nu, k, \tau, \rho, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ , then, under the conditions of Theorem 3.2 we obtain that,  $\mathbf{W} = \mathbf{Y}^* \mathbf{Y} \sim \mathcal{R}_n^{\beta, II}(m\beta/2, \tau, \mathbf{I}_n)$ , with  $\text{Re}(m\beta/2) > (n-1)\beta/2 + t_1$  and the density of  $\tilde{\mathbf{F}}_1$  is,

$$\propto |\tilde{\mathbf{F}}_1|^{(m-n+1)\beta/2-1} (1 + \rho \text{tr } \tilde{\mathbf{F}}_1)^{-[(mn+\nu)\beta/2-k-\sum_{i=1}^n t_i]} q_\tau^{-1}(\tilde{\mathbf{F}}_1)(d\tilde{\mathbf{F}}_1), \quad (25)$$

with constant of proportionality

$$\frac{\Gamma_1^\beta [(\nu + mn)\beta/2 - k - \sum_{i=1}^n t_i] \rho^{\beta mn/2 - \sum_{i=1}^n t_i}}{\Gamma_n^\beta[m\beta/2, -\tau] \Gamma_1^\beta[\nu\beta/2 - k]},$$

where  $\text{Re}(\nu\beta/2) > (n-1)\beta/2 + k_1$  and  $\text{Re}(m\beta/2) > (n-1)\beta/2 + t_1$ .  $\tilde{\mathbf{F}}_1$  is said to have a matrix multivariate k-beta-Riesz type II distribution.

*Proof.* Proofs are obtained from densities (22) and (23) by the following substitutions,

$$m \rightarrow n, \quad n \rightarrow m. \quad \square \quad (26)$$

$\square$

Suppose that  $\mathbf{A} \in \mathcal{L}_{m \times m}^\beta$  is any square root of constant matrix  $\mathbf{\Pi} = \mathbf{A}\mathbf{A}^* \in \mathfrak{P}_m^\beta$ . And, define  $\mathbf{Z} = \mathbf{A}^* \mathbf{F}_1 \mathbf{A}$ , therefore:

**Corollary 3.1.** 1. If  $\mathbf{F}_1$  has a matrix multivariate c-beta-Riesz type II distribution, then the density of  $\mathbf{Z}$  is,

$$\propto |\mathbf{Z}|^{(n-m+1)\beta/2-1} (1 + \rho \text{tr} \mathbf{\Pi}^{-1} \mathbf{Z})^{-[(mn+\nu)\beta/2+k+\sum_{i=1}^m t_i]} q_\tau(\mathbf{Z})(d\mathbf{Z}), \quad (27)$$

with constant of proportionality

$$\frac{\Gamma_1^\beta [(\nu + mn)\beta/2 + k + \sum_{i=1}^m t_i] \rho^{\beta mn/2 + \sum_{i=1}^m t_i}}{\Gamma_m^\beta[n\beta/2, \tau] \Gamma_1^\beta[\nu\beta/2 + k] q_\tau(\mathbf{\Pi}) |\mathbf{\Pi}|^{n\beta/2}},$$

where  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 - k_m$  and  $\text{Re}(m\beta/2) > (m-1)\beta/2 - t_m$ . Then  $\mathbf{Z}$  is said to have a nonstandardised matrix multivariate c-beta-Riesz type II distribution.

2. If  $\mathbf{F}_1$  has a matrix multivariate c-beta-Riesz type II distribution, then the density of  $\mathbf{Z}$  is,

$$\propto |\mathbf{Z}|^{(n-m+1)\beta/2-1} (1 + \rho \text{tr} \mathbf{\Pi}^{-1} \mathbf{Z})^{-[(mn+\nu)\beta/2-k-\sum_{i=1}^m t_i]} q_\tau^{-1}(\mathbf{Z})(d\mathbf{Z}), \quad (28)$$

with constant of proportionality

$$\frac{\Gamma_1^\beta [(\nu + mn)\beta/2 - k - \sum_{i=1}^m t_i] \rho^{\beta mn/2 - \sum_{i=1}^m t_i} q_\tau(\mathbf{\Pi})}{\Gamma_m^\beta[n\beta/2, -\tau] \Gamma_1^\beta[\nu\beta/2 - k] q_\tau(\mathbf{\Pi}) |\mathbf{\Pi}|^{n\beta/2}},$$

where  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 + k_1$  and  $\text{Re}(m\beta/2) > (m-1)\beta/2 + t_1$ . Then  $\mathbf{Z}$  is said to have a nonstandardised matrix multivariate k-beta-Riesz type II distribution.

*Proof.* The proof follows from (22) and (23) by applying (17) and (12).  $\square$   $\square$

If in theorems and corollaries in this section are defined  $k = 0$  and  $\tau = (0, \dots, 0)$ , the results in Díaz-García and Gutiérrez-Jáimez (2012) are obtained as particular cases. Also, in real case, when  $k = 0$  and  $\tau = (0, \dots, 0)$  the results in Corollary 3.1 contain as particular case the results in Muirhead (1982, Problem 3.18, p. 118).

## 4 Singular value densities

In this section, the joint densities of the singular values of matrices  $\mathbf{T}_1$ , and  $\tilde{\mathbf{T}}_1$  types I and II are derived. In addition, and as a direct consequence, the joint densities of the eigenvalues of  $\mathbf{F}_1$ , and  $\tilde{\mathbf{F}}_1$  types I and II are obtained for real normed division algebras.

**Theorem 4.1.** 1. Let  $\alpha_1, \dots, \alpha_m$ ,  $\alpha_1 > \dots > \alpha_m > 0$ , be the singular values of the random matrix  $\mathbf{T}_1 \sim \mathcal{MTR}_{m \times n}^{\beta, I}(\nu, k, \tau, \rho, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ . Then its joint density is

$$\propto \prod_{i=1}^m \alpha_i^{(n-m+1)\beta+2t_i-1} \left( 1 + \rho \sum_{i=1}^m \alpha_i^2 \right)^{-(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i} \prod_{i < j}^m (\alpha_i^2 - \alpha_j^2)^\beta \quad (29)$$

where the constant of proportionality is

$$\frac{2^m \pi^{\beta m^2/2 + \tau} \Gamma_1^\beta [(\nu + mn)\beta/2 + k + \sum_{i=1}^m t_i] \rho^{\beta mn/2 + \sum_{i=1}^m t_i}}{\Gamma_m^\beta[\beta m/2] \Gamma_m^\beta[n\beta/2, \tau] \Gamma_1^\beta[\nu\beta/2 + k]},$$

2. Let  $\alpha_1, \dots, \alpha_m$ ,  $\alpha_1 > \dots > \alpha_m > 0$ , be the singular values of the random matrix  $\mathbf{T}_1 \sim \mathcal{MTR}_{m \times n}^{\beta, II}(\nu, k, \tau, \rho, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ . Then its joint density is

$$\propto \prod_{i=1}^m \alpha_i^{(n-m+1)\beta-2t_i-1} \left( 1 + \rho \sum_{i=1}^m \alpha_i^2 \right)^{-(\nu+mn)\beta/2-k-\sum_{i=1}^m t_i} \prod_{i < j}^m (\alpha_i^2 - \alpha_j^2)^\beta \quad (30)$$

where the constant of proportionality is

$$\frac{2^m \pi^{\beta m^2/2 + \tau} \Gamma_1^\beta [(\nu + mn)\beta/2 - k - \sum_{i=1}^m t_i] \rho^{\beta mn/2 - \sum_{i=1}^m t_i}}{\Gamma_m^\beta[\beta m/2] \Gamma_m^\beta[n\beta/2, -\tau] \Gamma_1^\beta[\nu\beta/2 - k]},$$

Where  $\tau$  is defined in Lemma 2.3.

*Proof.* This follows immediately from (20) and (21) respectively, first using (18) and then applying (2).  $\square$

Analogously, joint densities of the singular values of  $\tilde{\mathbf{T}}_1$  types I and II are obtained from (29) and (30), making the substitutions (26).

Finally, observe that  $\alpha_i = \sqrt{\text{eig}_i(\mathbf{T}_1 \mathbf{T}_1^*)}$ , where  $\text{eig}_i(\mathbf{A})$ ,  $i = 1, \dots, m$ , denotes the  $i$ -th eigenvalue of  $\mathbf{A}$ . Let  $\gamma_i = \text{eig}_i(\mathbf{T}_1 \mathbf{T}_1^*) = \text{eig}_i(\mathbf{F}_1)$ , observing that, for example,  $\alpha_i = \sqrt{\gamma_i}$ . Then

$$\bigwedge_{i=1}^m d\alpha_i = \bigwedge_{i=1}^m 2^{-m} \prod_{i=1}^m \gamma_i^{-1/2} d\gamma_i,$$

the corresponding joint densities of  $\gamma_1, \dots, \gamma_m$ ,  $\gamma_1 > \dots > \gamma_m > 0$  types I and II, are obtained from (29) and (30) respectively as

$$1. \quad \propto \prod_{i=1}^m \gamma_i^{(n-m+1)\beta/2+t_i-1} \left( 1 + \rho \sum_{i=1}^m \gamma_i \right)^{-(\nu+mn)\beta/2+k+\sum_{i=1}^m t_i} \prod_{i < j}^m (\gamma_i - \gamma_j)^\beta$$

where the constant of proportionality is

$$\frac{\pi^{\beta m^2/2 + \tau} \Gamma_1^\beta [(\nu + mn)\beta/2 + k + \sum_{i=1}^m t_i] \rho^{\beta mn/2 + \sum_{i=1}^m t_i}}{\Gamma_m^\beta[\beta m/2] \Gamma_m^\beta[n\beta/2, \tau] \Gamma_1^\beta[\nu\beta/2 + k]},$$

2.

$$\propto \prod_{i=1}^m \gamma_i^{(n-m+1)\beta/2-t_i-1} \left(1 + \rho \sum_{i=1}^m \gamma_i\right)^{-[(\nu+mn)\beta/2-k-\sum_{i=1}^m t_i]} \prod_{i<j}^m (\gamma_i - \gamma_j)^\beta$$

where the constant of proportionality is

$$\frac{\pi^{\beta m^2/2+\tau} \Gamma_1^\beta [(\nu+mn)\beta/2-k-\sum_{i=1}^m t_i] \rho^{\beta mn/2-\sum_{i=1}^m t_i}}{\Gamma_m^\beta [\beta m/2] \Gamma_m^\beta [n\beta/2, -\tau] \Gamma_1^\beta [\nu\beta/2-k]},$$

## Conclusions

As in others works related, observe that the real dimension of real normed division algebras can be expressed as powers of 2,  $\beta = 2^n$  for  $n = 0, 1, 2, 3$ . On the other hand, as it can be corroborated in Kabe (1984), the results obtained in this work can be extended to the hypercomplex cases; that is, for complex, bicomplex, biquaternion and bioctonion (or sedenionic) algebras, which of course are not division algebras (except the complex algebra). Note also, that hypercomplex algebras are obtained by replacing the real numbers with complex numbers in the construction of real normed division algebras. Thus, the results for hypercomplex algebras are obtained by simply replacing  $\beta$  with  $2\beta$  in our results. Alternatively, following Kabe (1984), we can conclude that, our results are true for ‘ $2^n$ -ions’,  $n = 0, 1, 2, 3, 4, 5$ , emphasising that only for  $n = 0, 1, 2, 3$  the corresponding algebras are real normed division algebras.

The interest in these generalisations from a theoretical point of view becomes imminent, but from the practical point of view, we must keep in mind the fact from Baez (2002), *there is still no proof that the octonions are useful for understanding the real world*. We can only hope that eventually this question will be settled on one way or another. Also, for the sake of completeness, in the present article the case of octonions is considered, but the veracity of the results obtained for this case can only be conjectured; since there are still many problems under study in the context of the octonions.

Finally, note that if in sections 3 and 4 is defined  $\tau = (p, \dots, p)$  the corresponding results for the matrix multivariate Kotz type distribution are obtained as particular case, see Fang and Li (1999).

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